# Finite-dimensional corrections to mean field in a short-range $p$-spin glassy model

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In this work we discuss a short range version of the $p$-spin model. The model is provided with a parameter that allows to control the crossover with the mean field behaviour. We detect a discrepancy between the perturbative approach and numerical simulation. We attribute it to non-perturbative effects due to the finite probability that each particular realization of the disorder allows for the formation of regions where the system is less frustrated and locally freezes at a higher temperature.

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## I. INTRODUCTION

The mean field approximation is certainly a very useful starting point to study systems with long range interactions. Many physical features may differ from mean field behaviour if a model with short range interactions in finite dimension is considered. Generally speaking, the analysis of the Gaussian fluctuations is the first step to understand if the mean field approximation is well suited to describe the physics of the model. The perturbative fluctuations around the minima of the effective action may change the critical behaviour of the model or even destroy the phase transition.

In spin glasses and other similar glassy models the situation is rather complicated [1]. Generally speaking, one can distinguish two main classes of models: a first class, which comprehends models that in mean field undergo a continuous phase transition with a full replica symmetry breaking, and a second class, comprehending models which are well described, in mean field, by a 1RSB low-temperature solution. For this latter class, if the replica symmetry is broken with a parameter $q_1$ that is of order $O(1)$, the transition is discontinuous. Models of the first class describe well the physics of real spin glasses which undergo a second order phase transition with a divergent non-linear susceptibility and long range correlations. For these models the analysis of the infrared divergences of the propagators is rather subtle because not all of the modes in replica space are simply related to fluctuations of a physical quantity [2,3].

In this work we are studying models of the second class where the transition does not

imply any zero mass mode. The transition is discontinuous in the sense that, for $T < T_c$, another solution, different from the high temperature one, becomes dominant in the partition function for large $N$. In this kind of transition there are no precursor effects like quantities that diverge as $T$ approaches $T_c$ from above. Physically speaking, the discontinuity of the transition reflects the fact that the number of the metastable states becomes non-extensive for low enough temperatures and so a finite probability that two replicas belong to the same state arises. The probability $P(q)$ that two replicas $a$ and $b$ have an overlap $q_{ab} = q$ develops a delta function at $q_1$ when $T < T_c$. The weight of this delta function goes to zero as $T$ goes to $T_c$ and it increases continuously as the temperature decreases below $T_c$. This implies that thermodynamic quantities such as the entropy are continuous at $T_c$ and no latent heat is involved in the transition which therefore appears to be half way between first and second order. One may wonder if the strange features of this transition survive when models of the second class are considered in finite dimensions.

In this work we shall introduce a short range version of the $p$-spin glass model. The model, defined in $d$ spatial dimensions, contains an appropriate parameter $M$ ensuring the mean field solution to be exact for $M \to \infty$.

The analysis of the perturbative fluctuations (which are of order $O(1/M)$) around the MF solution and of the Gaussian propagators provides some insight on the nature of the transition in finite dimension. A second order phase transition would imply the existence of zero mass modes at the critical temperature which reflect the divergence of the correlation length. We will find no divergence of the zero momentum propagators and therefore this transition seems to invoke no diverging correlations. Moreover, calculating the $O(1/M)$ corrections of the relevant thermodynamic quantities around the saddle point, we observe that a perturbative approach shows no qualitative change of the phase transition with respect to mean field: the transition still appears discontinuous and still involves no latent heat.

Numerical simulations on the model [6–8] nevertheless indicate the existence, in $d = 3$ and in $d = 4$, of a divergent susceptibility and of all the phenomenology typical of a continuous second order transition at a temperature greater or equal than the mean field static critical temperature.

This apparent conflict between the two results may possibly be explained considering non-perturbative effects: the existence of regions in space where the system is locally less frustrated than on average and in which the spins are long range-correlated even for some $T > T_c$.

The paper is organized as follows. In the next section we shall briefly review the replica solution of the long-range $p$-spin glass model in absence of external magnetic field. We will refer to the original references for a more detailed discussion of the model and of the replica method. In section III we will introduce our short-range version of the model. In section IV we will calculate the Gaussian fluctuations around the large-$M$ solution of the model in the hot and in the cold phase. In section V we will calculate the propagators which show no zero-mass modes. We will also discuss the propagators on the dynamical line *i.e.* obtained by imposing the condition of marginality. Finally, we shall draw a possible interpretation of the physical behaviour of the model.

## II. THE LONG RANGE $P$-SPIN MODEL

The long range $p$-spin glass model is defined by the following Hamiltonian

$$\mathcal{H}_p(\{s\}) = -\sum_{(1\leq i_1<i_2<\cdots<i_p\leq N)} J_{i_1,i_2,\cdots,i_p} s_{i_1}\cdots s_{i_p} - h\sum_i s_i, \tag{1}$$

where $h$ is an external magnetic field and the interactions $J_{i_1,i_2,\cdots,i_p}$ are random variables distributed with the following Gaussian distribution

$$P(J_{i_1,i_2,\cdots,i_p}) = \left[\frac{N^{p-1}}{\pi J^2 p!}\right]^{-\frac{1}{2}} \exp\left[-\frac{(J_{i_1,i_2,\cdots,i_p})^2 N^{p-1}}{J^2 p!}\right]. \tag{2}$$

The scaling of the variance with $N^{p-1}$ ensures the free energy to be extensive. For $p = 2$ the interactions are the more familiar $J_{ij}$ interactions and the model is the well known SK model [9]. For $p > 2$ the model is generally addressed to as the '$p$-spin model'. The interactions between two spins $s_k$ and $s_l$ depend on all the $J_{i_1,i_2,\cdots,i_p}$ having $k$ and $l$ as one of the arguments as well as all the remaining spin variables of the system.

The behaviour of the model for $p = 2$ and for $p > 2$ is quite different at low temperatures. In this work we shall focus on the physics of the $p$-spin for $p > 2$. Note that (1) describes a model which is intrinsically mean field for large $N$ since all the $N$ spin variables interact reciprocally and there is no geometry in space.

To solve the model, one can introduce $n$ replicas of the system and calculate the replicated partition function, [15]

$$\overline{Z^n} = \int \prod \delta J_{i_1,\cdots,i_p} P(J_{i_1,\cdots,i_p}) \sum_{\{s_i^a\}} \left[\exp\beta\sum_{a=1}^{n}\left[\sum_{i_1<\cdots<i_p} J_{i_1,\cdots,i_p} s_{i_1}^a\cdots s_{i_p}^a + h\sum_i s_i^a\right]\right]. \tag{3}$$

From now on we shall always set $J = 1$. Integrating over the disorder, and introducing $n(n-1)$ auxiliary fields $Q_{ab}$ and $\lambda_{ab}$ (the $Q_{ab}$ and $\lambda_{ab}$ are symmetric $n\times n$ matrices with zero diagonal) one obtains the following expression

$$\overline{Z^n} = e^{nN\beta^2/4}\int_{-\infty}^{\infty}\prod_{a<b} dQ_{ab}\int_{-i\infty}^{i\infty}\prod_{a<b}\frac{d\lambda_{ab}}{2\pi}\exp\left[-NG(Q_{ab},\lambda_{ab})\right], \tag{4}$$

with

$$G(Q_{ab},\lambda_{ab}) = -\frac{\beta^2}{4}\sum_{a\neq b} Q_{ab}^p + \frac{1}{2}\sum_{a\neq b}\lambda_{ab}Q_{ab} - \ln Z[\lambda], \tag{5}$$

where

$$Z[\lambda] = Tr_{\{s_a\}}\exp\left[\frac{1}{2}\sum_{a\neq b}\lambda_{ab}s_a s_b + \beta h\sum_a s_a\right]. \tag{6}$$

In equation (6) the spin variables have lost their dependency from the index $i_r$. The fields $\lambda_{ab}$ have been introduced as Lagrange multipliers to impose the conditions

$$Q_{ab} = \frac{1}{N}\sum_i^N s_i^a s_i^b. \tag{7}$$

The saddle point of the functional integral (4) gives the mean field equations for the fields. Condition (7) gives physical meaning to the solution $Q_{ab}$ that extremizes $G(Q_{ab}, \lambda_{ab})$. The saddle point equations for the $n(n-1)$ fields are

$$
\begin{aligned}
\lambda_{ab} &= \frac{\beta^2 p Q_{ab}^{p-1}}{2}, \\
Q_{ab} &= \overline{< S_a S_b >} = < S_a S_b >_{H(Q,S)},
\end{aligned}
\tag{8}
$$

where $< \cdot >_{H(Q,S)}$ stands for the average taken with the measure $\exp[-\beta H(Q,S)]$.

If we indicate with $Q_{ab}^{sp}$ and $\lambda_{ab}^{sp}$ the solutions of equations (8) the saddle point free energy of the system is given by the following formula

$$
F(\beta) = -N\frac{\beta}{4} + \lim_{n\to 0} \frac{N}{\beta n} G(Q_{sp}, \lambda_{sp}). \tag{9}
$$

### A. Replica Symmetric Solution

In the high temperature phase one chooses the RS ansatz for the saddle point matrix, so one has

$$
\begin{aligned}
Q_{ab} &= q, \qquad \text{for } a \neq b, \\
\lambda_{ab} &= \lambda \qquad \text{for } a \neq b, \\
Q_{aa} &= \lambda_{aa} = 0.
\end{aligned}
\tag{10}
$$

The saddle point equations deriving from this ansatz are

$$
\begin{aligned}
\lambda &= \frac{\beta^2 p q^{p-1}}{2}, \\
q &= \int Dz \tanh^2(z\sqrt{\lambda} + \beta h),
\end{aligned}
\tag{11}
$$

where we used the following notation

$$
\int Dz = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz. \tag{12}
$$

In the high temperature phase one has that $q = 0$ if $h = 0$. We shall always consider the $h = 0$ case for simplicity of notations and because we shall not be concerned with the effects of a magnetic field.

In this case one finds

$$
F_{RS} = -N\left[\frac{\beta}{4} + \frac{\ln 2}{\beta}\right]. \tag{13}
$$

This value of the free energy coincides with the 'annealed' result *i.e.* what would be obtained by calculating $\ln \overline{Z}$. From (13) one can derive the following expression for the entropy

$$
S_{RS} = N\left[\ln 2 - \frac{\beta^2}{4}\right]. \tag{14}
$$

The entropy in (II A) becomes negative for $T < 1/2\sqrt{\ln 2} \stackrel{\rm def}{=} T_c^*$, so at a greater or equal temperature $T_c(p)$ the replica symmetric solution will stop to be correct. There will be another solution dominating $\overline{Z^n}$ in the zero-$n$ limit, and this will involve the breaking of the original symmetry among the replicas. The RS solution though, will not become unstable (if $p > 2$) at $T_c(p)$ so the new solution shall be distant from the old one and the transition will therefore be discontinuous in the order parameter.

### B. 1RSB Solution

Here we show the 1RSB solution for the $p$-spin model for general $p$. For $p > 2$ this solution is correct below $T_c(p)$. For finite $p$ there is a second critical temperature $T^*(p)$ at which the system undergoes to a continuous full replica symmetry breaking. Here we will not study this second transition. So one has, for $p > 2$,

$$0 < T^*(p) < T_c(p) < 1.$$

In the large-$p$ limit one has that $T^*(\infty) \to 0$ and $T_c(\infty) \to 1/(2\sqrt{\ln 2})$. The 1RSB solution gives the following expression for $G_{sp}$

$$\lim_{n\to 0} \frac{G_{sp}}{n} = \frac{\beta^2}{4}\left(mq_0^p + (1-m)q_1^p\right) - \frac{1}{2}\left(mq_0\lambda_0 + (1-m)q_1\lambda_1\right) + \frac{\lambda_1}{2}$$
$$-\frac{1}{m}\int D(z)\ln\int D(y)(2\cosh[\sqrt{\lambda_0}z + \sqrt{\lambda_1-\lambda_0}y])^m, \tag{15}$$

and the saddle point equations

$$\begin{aligned}
\lambda_i &= \frac{\beta^2 p q_i^{p-1}}{2} \\
q_1 &= \int Dz\int Dy \tanh^2(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0})\cosh^m(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0}) \\
m & \frac{\beta^2}{4}(q_1^p - q_0^p)(1-p) = \frac{1}{m}\int Dz \ln\int Dy\cosh^m(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0}) - \\
&\int Dz \frac{\int Dy \cosh^m(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0})\ln\cosh(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0})}{\int Dy\cosh^m(z\sqrt{\lambda_0} + y\sqrt{\lambda_1-\lambda_0})}.
\end{aligned} \tag{16}$$

In absence of external magnetic field $q_0 = 0$. $m$ is an additional parameter which characterizes the form of the solution. The saddle point solution $m_{sp}(T)$ has a physical meaning in the replica method. Two different replicas have an overlap $q_0$ with probability $m_{sp}(T)$ and overlap $q_1$ with probability $1 - m_{sp}(T)$. For $0 < m_{sp}(T) < 1$ the $1RSB$ solution is the correct one and the critical temperature is given by the equation

$$m_{sp}(T_c) = 1.$$

In the SK model ($p = 2$) the RS solution becomes unstable at a higher temperature ($T_c^{SK} = 1$ with our normalizations) and there is a transition into a full $RSB$ phase with a

continuous solution $q(x)$ which holds for all temperatures below $T_c^{SK} = 1$. In this case the 1RSB is anyhow a better approximation than the RS one.

In figure (1) we plotted the free energy of the model for $p = 4$ for low temperatures. The high and low temperature curves are tangent at the critical temperature. In figures (2,3) we plotted the values of $q_1$ and $m$ obtained by the $1RSB$ solution for $p = 4$ in function of the temperature.

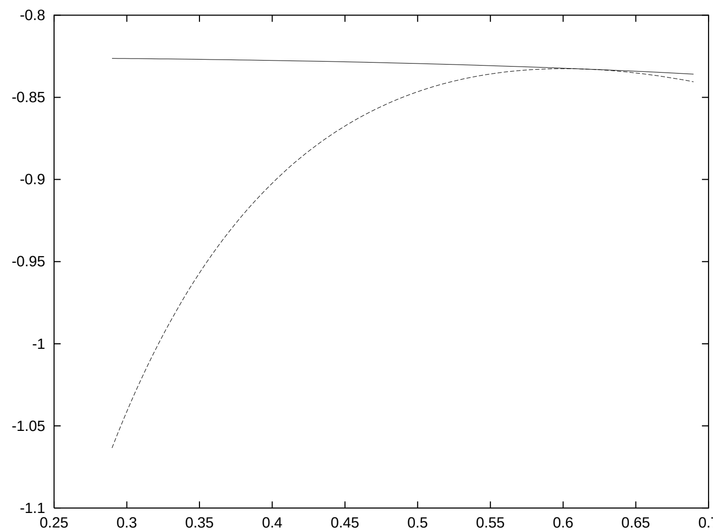


FIG. 1. Free energy of the $p$-spin model vs temperature for $p = 4$. The dashed line is the RS solution which is wrong at low temperatures.

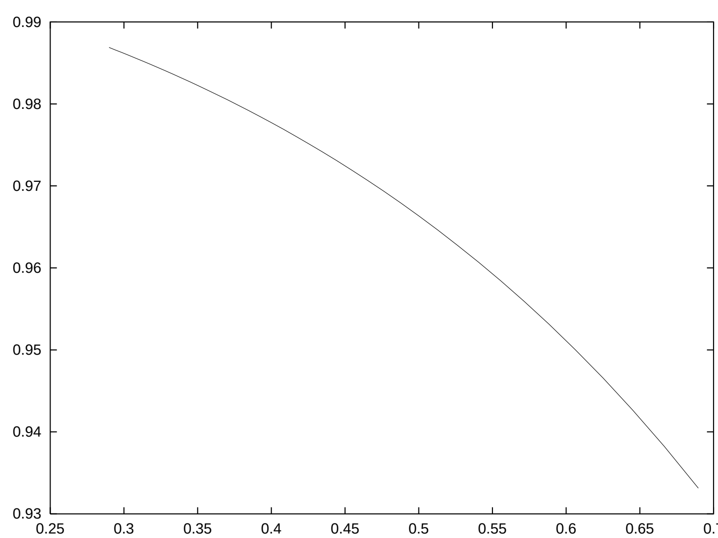


FIG. 2. $q_1$ vs temperature for $p = 4$.

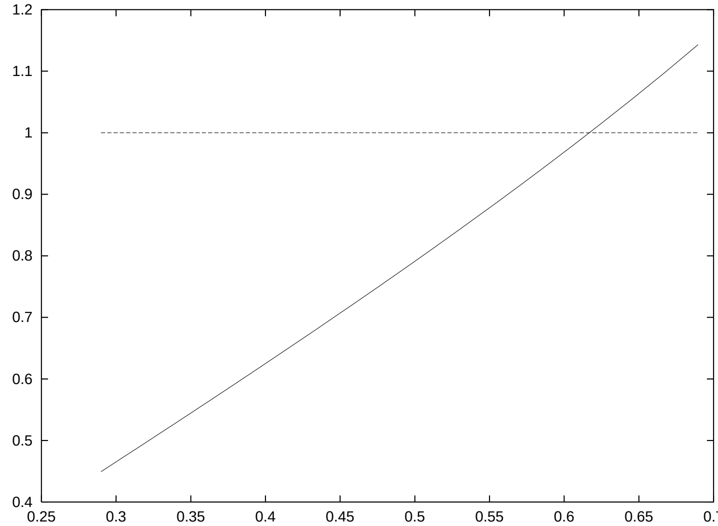

FIG. 3. $m$ vs temperature for $p = 4$. At the critical temperature the curve crosses the line $m = 1$.

## III. THE SHORT-RANGE MODEL

Here we introduce and study a short-range version of the $p$-spin glass. The model that we introduce is defined on an hyper-cubic $d$-dimensional lattice of side $L$. On each side of the lattice there are $M$ spins. Every spin interacts via quenched random couplings with $p-1$ spins chosen among spins on the same site and on nearest-neighbour sites. It is natural to expect (and it will also derive from the equations) that, for large $M$, one recovers the mean field solution since each spin interacts with a large number of other spins. The Hamiltonian of the model is

$$H(\{J\}) = \sum_{<l_1,\cdots,l_p>}^{L^d} \sum_{i_1,\cdots,i_p}^{M} J_{i_1\ldots i_p}^{l_1\ldots l_p} s_{i_1}(l_1)\cdots s_{i_p}(l_p). \tag{17}$$

By $\sum_{<l_1,\cdots,l_p>}^{L^d}$ we sum over all the sites of the lattice taking, for each couple of adjacent sites $i$ and $j$, $p-k$ of the $l_1,\cdots,l_p$ indices equal to $i$ and $k$ indices equal to $j$. In other words, for each nearest neighbour sites $i$ and $j$, every interaction involves $p-k$ spins of site $i$ and $k$ spins of site $j$ with $k$ running from zero to $p$. We consider discrete ($\pm 1$) spin variables and we call $s_{i_r}(l_r)$ the $i_r^{th}$ spin of site $l_r$ with $i_r$ running from 1 to $M$. The $J_{i_1\ldots i_p}^{l_1\ldots l_p}$ are quenched random variables extracted from the distribution

$$P(J_{i_1\ldots i_p}^{l_1\ldots l_p}) = \sqrt{\frac{aM^{p-1}}{\pi}}\exp\left[-aM^{p-1}(J_{i_1\ldots i_p}^{l_1\ldots l_p})^2\right], \tag{18}$$

where the normalization ensures an extensive free energy and the constant $a$ will be fixed later on by the condition that, in the large-$M$ limit, the free energy density of this model coincides with the one of the long range model.

The model just defined is a short range model, but the existence of the parameter $M$ ensures the mean field solution to be correct in the limit of $M \to \infty$. In this way one has

a systematic way to reduce arbitrarily the corrections to mean field which are due to the finiteness of the coordination number and one can control the crossover effects from finite-dimensional to mean field behaviour. A similar generalization of the Random Energy Model has been studied in the one dimensional case in [10].

The mean field solution of the model is identical to the solution of the long range $p$-spin model. Nevertheless, even in the large-$M$ limit, this model keeps an important ingredient of difference with respect to the long-range $p$-spin model because there is a geometry in space.

The expression for the partition function of $n$ replicas of the system is

$$\overline{Z^n} = e^{nML^d\beta^2/4} \prod_{a<b}^{n} \prod_{l}^{L} \int dQ_{ab}(l) d\lambda_{ab}(l) \exp\left[-MG[Q,\lambda]\right], \tag{19}$$

with

$$G[Q,\lambda] = -\frac{\beta^2}{2a} \sum_{a<b}^{n} \sum_{<l_1,\cdots,l_p>}^{L^d} \left(Q_{ab}^{l_1} \ldots Q_{ab}^{l_p}\right)^p + \sum_{a<b}^{n} \sum_{l}^{L^d} \lambda_{ab}^l Q_{ab}^l - \ln \mathrm{Tr} \exp\left[\sum_{a<b}^{n} \sum_{l}^{L^d} \lambda_{ab}^l s^a(l) s^b(l)\right], \tag{20}$$

We can write the non local part of the $G[Q,\lambda]$ in a more convenient form exploiting the following identity

$$\sum_{<l_1,\cdots,l_p>}^{L^d} \left(Q^{l_1} \ldots Q^{l_p}\right)^p = \sum_{l,m}^{L^d} K(l,m) \left(Q^l + Q^m\right)^p - (2^p + 2d - 1) \sum_{l}^{L^d} (Q^l)^p, \tag{21}$$

where

$$K(l,m) = \delta_{l,m} + \sum_{\vec{1}} \delta_{l+\vec{1},m},$$

and $\sum_{\vec{1}}$ is the sum over all possible directions of the unitary lattice spacing vector $\vec{1}$. So the function $G[Q,\lambda]$ takes the form

$$G[Q,\lambda] = -\frac{\beta^2}{2a} \sum_{a<b}^{n} \sum_{l,m}^{L^d} K(l,m) \left(Q_{ab}^l + Q_{ab}^m\right)^p - (2^p + 2d - 1) \sum_{l}^{L^d} (Q_{ab}^l)^p + \sum_{a<b}^{n} \sum_{l}^{L^d} \lambda_{ab}^l Q_{ab}^l - \ln \mathrm{Tr} \exp\left[\sum_{a<b}^{n} \sum_{l}^{L^d} \lambda_{ab}^l s^a(l) s^b(l)\right]. \tag{22}$$

The mean field solution must be constant in space because the first term in (20) can be re-written as a gradient plus a local term. We want to normalize the $G[Q,\lambda]$ in such a way to recover (5) in the mean field approximation. To do so one has to set

$$a = 2^p d - 2d + 1.$$

The saddle point equations yield

$$\lambda^l_{ab} = \frac{\beta^2 p}{2}(Q^l_{ab})^{p-1}$$
$$Q^l_{ab} = \left\langle S^a(l)S^b(l)\right\rangle_\lambda . \tag{23}$$

These equations are the same equations that we found in the long range case and are exact in the limit of $M \to \infty$. Therefore, in the large-$M$ limit, the equations (23) reduce to equations (11) or (16) depending on the temperature.

## IV. GAUSSIAN FLUCTUATIONS

For finite $M$ the saddle point equations are not exact. In this section we estimate the $O(1/M)$ corrections to the free energy due to the small fluctuations around the saddle point of both the high and low temperature phase. For finite $M$ we can write

$$Q_{ab} = Q^{sp}_{ab} + \delta Q^l_{ab} \qquad\qquad \lambda_{ab} = \lambda^{sp}_{ab} + \delta\lambda^l_{ab}, \tag{24}$$

where $\delta Q^l_{ab}$ and $\delta\lambda^l_{ab}$ are $O(1/M)$ quantities. $Q^{sp}_{ab}$ and $\lambda^{sp}_{ab}$ are the solutions of the equation (23) and are site independent.
It is convenient to work in Fourier space so we write

$$\delta Q^l_{ab} = \int_{-\pi}^{\pi} \frac{d^d\vec{k}}{(2\pi)^d}\delta Q_{ab}(k)e^{i\vec{k}\cdot\vec{l}}. \tag{25}$$

One can expand $G[Q,\lambda]$ up to quadratic order in $\delta Q^l_{ab}$ and $\delta\lambda^l_{ab}$. Observing that the couple $(\alpha\beta)$ takes $n(n-1)/2$ different values, we can write

$$\overline{Z^n} = e^{-M\beta F_{sp}} \prod_{\alpha=1}^{n(n-1)} \int \delta\Omega_\alpha \exp\left[-\frac{M}{2}\sum_{\alpha<\beta}^{n(n-1)} \int_{-\pi}^{\pi} \frac{d^d\vec{k}}{(2\pi)^d}\Omega_\alpha(\vec{k})\hat{M}_{\alpha\beta}\Omega_\beta(-\vec{k})\right], \tag{26}$$

where $MF_{sp}$ is the free energy of the system in the mean field approximation. We have used the following labeling:

$$\Omega_\alpha(\vec{k}) = \delta\lambda_{ab}(\vec{k}) \qquad \text{for} \quad 1 \le \alpha \le \frac{n(n-1)}{2}$$
$$\Omega_\alpha(\vec{k}) = \delta Q_{ab}(\vec{k}) \qquad \text{for} \quad \frac{n(n-1)}{2} < \alpha \le n(n-1). \tag{27}$$

$\hat{M}_{\alpha\beta}$ is the $n(n-1)\times n(n-1)$ matrix of the fluctuations

$$\hat{M} = \begin{pmatrix} M^{\lambda,\lambda}_{abcd} & M^{q,\lambda}_{abcd} \\ M^{q,\lambda}_{abcd} & M^{q,q}_{abcd} \end{pmatrix}, \tag{28}$$

with

$$M^{q,q}_{abcd} = -\frac{\beta^2 p(p-1)}{2}Q^{sp(p-2)}_{ab}\delta_{ab,cd}\frac{1-2d+2^p\cos^2\left(\frac{\mathbf{k}}{2}\right)}{1-2d+2^p d}$$
$$M^{q,\lambda}_{abcd} = \delta_{ab,cd},$$
$$M^{\lambda,\lambda}_{abcd} = -\left[\left\langle S^aS^bS^cS^d\right\rangle_\lambda - \left\langle S^aS^b\right\rangle_\lambda\left\langle S^cS^d\right\rangle_\lambda\right] \equiv -C_{abcd}. \tag{29}$$

We indicated with $C_{abcd}$ the connected four-spin correlation function. We also used the notation

$$\cos^2\left(\frac{\mathbf{k}}{2}\right) \equiv \cos^2\left(\frac{k_1}{2}\right) + \ldots + \cos^2\left(\frac{k_d}{2}\right).$$

Performing the Gaussian integral one obtains

$$F(T) = F_{sp}(T) + \frac{1}{M}\Delta F(T), \tag{30}$$

with

$$\Delta F = \frac{1}{2\beta}\lim_{n\to 0}\sum_{\lambda} m_\lambda \int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\ln\left[\lambda(\mathbf{k})\right], \tag{31}$$

where $m_\lambda$ is the multiplicity of the $\lambda(\vec{k})$ eigenvalue of $\hat{M}_{\alpha\beta}$.

If the temperature is above the critical temperature the mean field solution is RS with $q_0 = 0$ (we shall always consider the case of zero magnetic field). The matrix of the fluctuations has got the whole block $M^{q,q}_{abcd}$ equal to zero and it is straightforward to show that above $T_c$ the temperature-dependent part of the $O(1/M)$ corrections is zero and

$$\Delta F = 0. \tag{32}$$

In the low temperature phase the structure of $\hat{M}_{\alpha\beta}$ changes because the sub-matrix $M^{q,q}_{abcd}$ is not identical to zero but takes the following form

$$\begin{aligned} &M^{q,q}_{abcd} = 0 \text{ if } a \text{ and } b \text{ do not belong to the same block}\\ &M^{q,q}_{abcd} = f(q,\mathbf{k}) \text{ if } a \text{ and } b \text{ belong to the same block },\end{aligned}$$

where

$$f(q,\mathbf{k}) = -\frac{\beta^2 p(p-1)}{2} q^{p-2}\frac{1-2d+2^p\cos^2\left(\frac{\mathbf{k}}{2}\right)}{1-2d+2^p d}. \tag{33}$$

In this case one needs to calculate all the $n(n-1)$ eigenvalues of $M_{\alpha,\beta}$. The calculation is similar to the one performed in [12] in the case of the Little model with the difference that in that case the model was Gaussian and there was no need to introduce the $\lambda_{\alpha\beta}$ field. So here the eigenvalues are $n(n-1)$. We shall indicate with $\lambda_\Lambda$ and $\lambda_Q$ the eigenvalues of the two sub-blocks $M^{\lambda,\lambda}_{abcd}$ and $M^{q,q}_{abcd}$. It is easy to calculate the determinant of the total matrix $\hat{M}$ observing that, because of the diagonal structure of $M^{q,q}_{abcd}$, correspondingly to every $\lambda_\Lambda$ and $\lambda_Q$ there are two eigenvalues of the total matrix

$$\lambda_\pm = \frac{\lambda_\Lambda + \lambda_Q \pm\sqrt{(\lambda_\Lambda - \lambda_Q)^2 + 4}}{2}, \tag{34}$$

so that $\lambda_+\lambda_- = \lambda_\Lambda\lambda_Q - 1$.

The structure of the spectrum of the eigenvalues $\lambda_\Lambda$ and $\lambda_Q$ in the 1RSB phase is briefly reported in the appendix. The final expression for the $O(\frac{1}{M})$ corrections below $T_c$ becomes

$$\Delta F = \frac{1}{2\beta}\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\left\{\frac{(m-3)}{2}\ln\left[1+f(q,\mathbf{k})\left(C_{[abcd]}+(1-q)^2\right)\right]\right.$$
$$\frac{1}{m}\ln\left[1-f(q,\mathbf{k})\left(\frac{A+D+|A-D|}{2}\right)\right]$$
$$\left.\frac{m-1}{m}\ln\left[1-f(q,\mathbf{k})\left(\frac{A'+D'+|A'-D'|}{2}\right)\right]\right\}, \tag{35}$$

where we have defined

$$C_{[abcd]} = \left\langle S^a S^b S^c S^d\right\rangle_\lambda - \left\langle S^a S^b\right\rangle_\lambda \left\langle S^c S^d\right\rangle_\lambda ,$$

with the four replicas $a, b, c, d$ belonging to the same diagonal block of the 1RSB matrix. The constants $A, D, A', D'$ in equation (35) are also defined in the appendix.

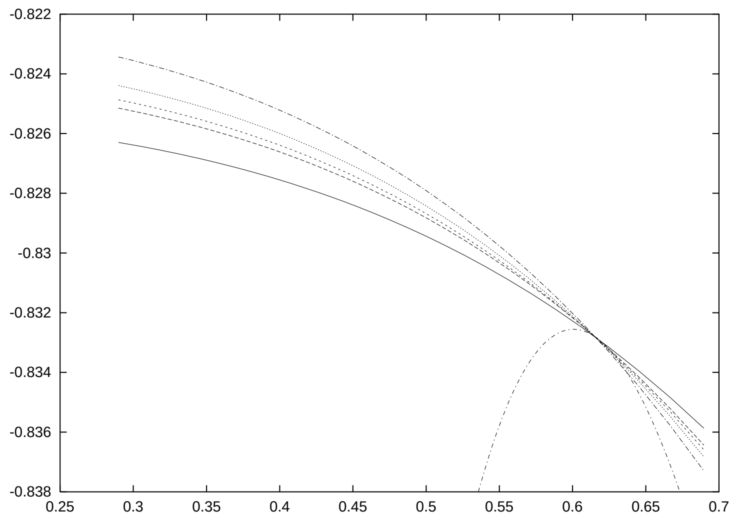


FIG. 4. Free energy density of the three-dimensional $p$-spin model vs temperature for $p = 4$. and $M = 4, 6, 8, 10, \infty$

As a check one can easily verify that, for $m = 1$, the result (32) is, as it should be, recovered since the structure of the equations becomes identical to the RS case after the identification $q_1 = q_0$.

To perform the above integrals in momentum space we make use of the identity

$$\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\ln\left[1+A\cos^2(\frac{\mathbf{k}}{2})\right] = -\int_0^\infty \frac{dt}{t}\exp[-t]\left[I_0^d\left(\frac{At}{2}\right)E^{-\left(\frac{Adt}{2}\right)}-1\right], \tag{36}$$

where $I_0$ is the modified Bessel function of order 0.

One can solve numerically the saddle point equations and then plug the value of $q_1(T), m(T)$ into the expression of $G$ and $\Delta G$. The result, for $d = 3$ and $M = 4, 6, 8, 10, \infty$, is shown in figure (4). In the plot the higher are the curves the smaller is the value of $M$. We can see that the perturbative corrections slightly shift the curves but do not change the nature of the transition. The critical temperature, defined as the temperature where the

curves of the high and low temperature free energy coincide, is the same at which $m_{sp} = 1$. Furthermore, the two curves are tangent at the critical temperature denoting a zero latent heat as in the MF case. Finally, we also remark that the transition temperature changes very slightly with $M$. A rough estimate the $O(1/M)$ shift in the critical temperature gives

$$T_c(M) \simeq T_c - \frac{1}{M}(0.0168).$$

## V. REPLICA PROPAGATORS

### A. At equilibrium

The study of the Gaussian propagators provides knowledge on the existence of zero-mass modes and therefore on the nature of the transition. In this section we consider the corrections to the mean field solution of the theory defined by the (22). In a perturbative approach we define and calculate the propagators of the $p$-spin model in the RS and 1RSB phase.

The propagators $G_{abcd}$ are defined in the following way

$$\begin{aligned} G^{Q}_{abcd} &= \langle \delta Q_{ab} \delta Q_{cd} \rangle \\ G^{\Lambda Q}_{abcd} &\equiv G^{Q\Lambda}_{cdab} = \langle \delta \Lambda_{ab} \delta Q_{cd} \rangle \\ G^{\Lambda}_{abcd} &= \langle \delta \Lambda_{ab} \delta \Lambda_{cd} \rangle . \end{aligned}$$

The equations for the propagators are easily found by standard arguments

$$\langle \phi(x) \frac{\partial S[\phi]}{\partial \phi(y)} \rangle = \delta(x-y), \tag{37}$$

and in the case of our functional integral (19) they become

$$\begin{aligned} &\langle \delta Q_{ab} \frac{\partial G[Q,\Lambda]}{\partial Q_{cd}} \rangle = \delta_{ab,cd} \qquad && \langle \delta \Lambda_{ab} \frac{\partial G[Q,\Lambda]}{\partial Q_{cd}} \rangle = 0 \\ &\langle \delta Q_{ab} \frac{\partial G[Q,\Lambda]}{\partial \Lambda_{cd}} \rangle = 0 \qquad && \langle \delta \Lambda_{ab} \frac{\partial G[Q,\Lambda]}{\partial \Lambda_{cd}} \rangle = \delta_{ab,cd} . \end{aligned} \tag{38}$$

It is straightforward to obtain the following linear equations for the propagators

$$\begin{aligned} &2 f(Q_{cd}, \mathbf{k}) G^{Q}_{abcd} + 2 G^{\Lambda Q}_{abcd} = \delta_{ab,cd} \\ &2 f(Q_{cd}, \mathbf{k}) G^{\Lambda Q}_{abcd} + 2 G^{\Lambda}_{cdab} = 0 \\ &2 G^{Q}_{abcd} - 2 \sum_{r<s} C_{rscd} G^{\Lambda Q}_{rsab} = 0 \\ &2 G^{\Lambda Q}_{abcd} - 2 \sum_{r<s} C_{rscd} G^{\Lambda}_{rsab} = \delta_{ab,cd}, \end{aligned} \tag{39}$$

where $f(Q_{cd}, \mathbf{k})$ is defined by (33).

We shall always use the fact that, in absence of magnetic field, $q_0 = 0$. There are different kind of propagators depending on the relation between the replica indices they

refer to, so for different choices of indices $abcd$ in equations (39) we will obtain different equations. In the RS phase one obtains

$$G^{\Lambda}_{abcd} = 0 \tag{40}$$

$$G^{\Lambda Q}_{abcd} = \frac{1}{2}\delta_{ab,cd} \tag{41}$$

$$G^{Q}_{abcd} = \frac{1}{2}C_{abcd}, \tag{42}$$

where $C_{abcd}$ is the four-spin connected correlation function defined in (29). So in the replica-symmetric phase all propagators $G^{Q}_{abcd}$ are zero except for the diagonal one which is trivial

$$G^{Q}_{abab} = G^{Q}_{abba} = \frac{1}{2}. \tag{43}$$

In the 1RSB phase, we shall use the convention to indicate with $[ab]$ two replicas belonging to the same block and with $[a][b]$ two replicas belonging to different blocks of the 1RSB saddle point matrix $Q^{sp}_{ab}$. If $a$ and $b$ (or $c$ and $d$) do not belong to the same block , the form of the propagators is still given by equation (42) where the function $C_{abcd}$ changes in the low temperature phase so one has

$$\begin{aligned} G^{Q}_{01} &\equiv G^{Q}_{[a][b][a][b]} = \frac{1}{2}. \\ G^{Q}_{02} &\equiv G^{Q}_{[a][b][c][b]} = \frac{1}{2}q_1, \quad [ac]. \\ G^{Q}_{03} &\equiv G^{Q}_{[a][b][c][d]} = \frac{1}{2}q_1^2, \quad [ac] \quad [bd]. \end{aligned} \tag{44}$$

where $a$ and $b$ belong respectively to the same blocks as $c$ and $d$.

For the replica indices in the same block, $[ab]$, we have $Q_{ab} = q_1$ and from equations (39) we obtain the following equations for $G^{\Lambda}$ and $G^{Q}$

$$\begin{aligned} G^{Q}_{abcd} &= f(q_1,\mathbf{k})^{-1}\delta_{ab,cd} + f(q_1,\mathbf{k})^{-2}G^{\Lambda}_{abcd} \\ G^{\Lambda}_{abcd} &= -f(q_1,\mathbf{k})\sum_{r<s} C_{rscd}G^{\Lambda}_{rsab} - \frac{f(q_1,\mathbf{k})}{2}\delta_{ab,cd}. \end{aligned} \tag{45}$$

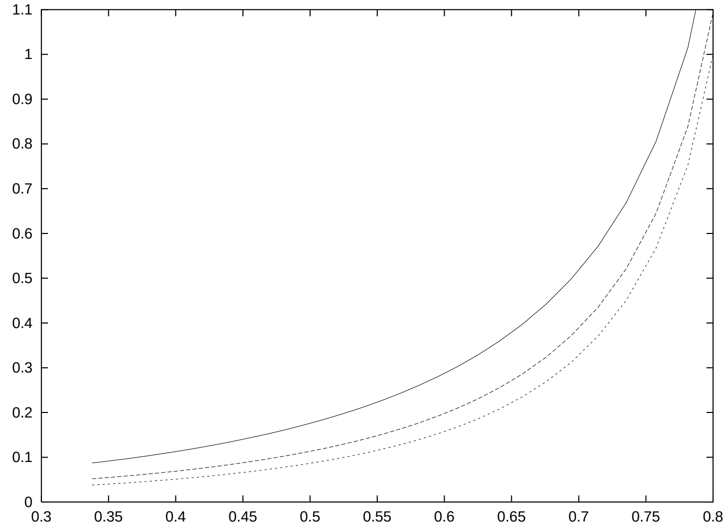

FIG. 5. $G_1^\Lambda$, $G_2^\Lambda$, $G_3^\Lambda$ for $p = 4$ and $d = 3$

The second of the equations (45) can be written for the various choices of the indices $abcd$ obtaining a set of four coupled equations for $G_1, G_2, G_3, G_4$ defined as follows:

$$
\begin{aligned}
G_1 &= G_{[abab]} \qquad & G_2 &= G_{[abar]} \\
G_3 &= G_{[abrs]} \qquad & G_4 &= G_{[ab][rs]}
\end{aligned}
\tag{46}
$$

Solving the equations and plugging in the MF values for $q_1$ and $m$ we obtain the propagators in function of the temperature. From the first of the equations (45) it is possible to define the propagators $G^Q_{abcd}$. One has that, in absence of magnetic field $G_4^\Lambda(k = 0) = 0$. In figure (5) we plot the $G_1^\Lambda, G_2^\Lambda, G_3^\Lambda$ for $k = 0$ versus the temperature. In conclusion, we observe no divergence of the RS and 1RSB propagators at the critical temperature.

This means that no zero-mass modes are present around the stable saddle point solutions and usually it indicates that no continuous transition is taking place.

### B. On the dynamical line

We recall that in the long range $p$-spin model, there is a dynamical critical temperature $T_d > T_c$ below which the system does not reach equilibrium in finite times. In the mean field approach, if we quench the system to low temperature coming from an high temperature state, the system goes to a dynamical metastable state, having an energy greater than the equilibrium one. In [13], it has been shown that if the system starts from a random initial configuration, it evolves following the flat directions connecting an ensemble of metastable states which are the threshold solutions to the TAP equations [14].

This cannot happen in short range models and it is and artifact of mean field theory. However it can be interpreted as the signal of starting a quite slow approach to equilibrium. In any case the disappearance of this metastable energy is a non perturbative result, so that it make sense to study the properties of the correlation functions in this state.

In the replica approach it can be shown that in the dynamical metastable state the value of $m$ is not fixed my extremizing the free energy, but by imposing the condition that the replicon eigenvalue is equal to zero. We are going to compute the propagators in this region as functions of the temperature. The formulae are similar to the previous ones, only the value of $m$ will be different.

This amounts to set

$$f(q, \mathbf{k} = \mathbf{0}) = \frac{1}{\lambda_\Lambda^R} \tag{47}$$

where $\lambda_\Lambda^R$ is defined in the appendix.

We thus obtain some new equations for the propagators concerning replicas belonging to the same block

$$G_1 = \frac{(6 - 5m + m^2)(q - r_0)(4q - mq^2 - 4r_0 + mr_0)}{A(f(q, \mathbf{k}) - l_0)(f(q, \mathbf{k}) - l_0 - (m-1)l_1)(f(q, \mathbf{k}) - l_2)} \tag{48}$$

$$G_2 = \frac{(3 - m)(q - r_0)(4q - mq^2 - 4r_0 + mr_0)}{A(f(q\mathbf{k},) - l_0)(f(q, \mathbf{k}) - l_0 - (m-1)l_1)(f(q\mathbf{k},) - l_2)} \tag{49}$$

$$G_3 = \frac{2(q - r_0)(4q - mq^2 - 4r_0 + mr_0)}{A(f(q, \mathbf{k}) - l_0)(f(q, \mathbf{k}) - l_0 - (m-1)l_1)(f(q\mathbf{k},) - l_2)} \tag{50}$$

where

$$\begin{aligned}
A &= (-1 + 2q - r_0)^2(-2 + 16q - 6mq - 32q^2 + 23m^2q - 3m^2q^2 + 4mq^3 - 5m^2q^3 + \\
&\quad m^3q^3 - 12r_0 + 7mr0 - m^2r_0 + 48qr_0 - 46mqr_0 + 13m^2qr_0 + \\
&\quad -m^3qr0 - 3mq^2r_0 + 4m^2q^2r_0 - m^3q^2r_0 - 18r_0^2 + 21mr_0^2 - 8m^2r_0^2 + m^3r_0^2) \\
l_0 &= \frac{-1}{1 - 2q + r_0} \\
l_1 &= \frac{1}{-1 + 4q - mq - 3r_0 + mr_0} \\
l_2 &= \frac{-2}{2 - 8q + 4mq + mq^2 - m^2q^2 + 6r_0 - 5mr_0 + m^2r_0}.
\end{aligned} \tag{51}$$

In the previous formulae, $r_0 = C_{[abcd]}$. In the general case the propagators diverge, for small $k$, as $k^{-2}$. The condition $m = 1$ now gives the dynamical critical temperature, and one can see that all the propagators in (50) coincide and that the divergence is of order $k^{-4}$.

The divergence at small momenta of the propagator is not unexpected [16]. Indeed the vanishing of the replicon propagator implies a divergence of the susceptibility and consequently a singularity at $k = 0$. The form of the singularity could not be predicted using general arguments. The change of the exponent at $T_d$ is particularly striking; it is related to the degeneracy of the replicon and longitudinal eigenvalue at $m = 1$.

Although the result was derived in the context of spin models, we believe that this structure of exponents $k^{-4}$ at $T_d$ and $k^{-2}$ at $T < T_d$ is quite general and it would be valid in many others models. It is also clear that these are mean field results: also if we remain in a perturbative framework, the exponents are likely to be changed for sufficient low dimension. The value of the upper critical dimension (6?, 8?), above which the mean fields exponents do not get perturbative corrections, can be extracted by analyzing the contribution of higher loops, but this task goes beyond the aim of this paper.

## VI. DISCUSSION

The model that we have described has been studied numerically in $d = 3$ with $p = 4$ and $M = 3, 4$ for $T > T_c$ [7] and below $T_c$ [6]. The results of the numerical simulations seem to indicate, the existence of a transition at a critical temperature $T_c$ from a high temperature phase to a broken replica symmetry phase. However, this transition appears to be of second order with divergent spin-glass susceptibility. A possible interpretation of this apparent contradictory phenomenology is the following [17]. For each realization of the disorder there are some regions of space in which the effect of frustrations are weaker than in the rest of the system. Within these regions the system is likely to freeze at a temperature $T_r$ which is higher than the temperature $T_c$ at which the whole system freezes. So for $T > T_c$ (not too high temperatures though!) there are regions in space which the system is locally frozen, the typical size of the region being a function of the temperature $R^{d_r}(T)$ where $d_r$ is the dimensionality of the region. Within these regions the system is very strongly correlated and the correlation length is of order $R(T)$. The total SG susceptibility is the integral over space of the local SG susceptibilities and the contribution of the regions in which the system is strongly correlated grows with $R(T)$ and diverges when $R(T)$ becomes of the size of the whole system. So, in this interpretation, the transition remains discontinuous within the regions of space where it occurs, i.e. the local overlap changes discontinuously from $q_0$ to $q_1$, and the continuous varying quantity is the typical size of the regions where the system is strongly correlated. This work wants to be a step forward towards the comprehension of spin glass models and of structural glasses in finite dimensions parallel to other attempts [6–8]. There are still many unclear things on the subject which is worth, according to us, for further studies.

## VII. ACKNOWLEDGMENTS

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## APPENDIX A:

In the 1RSB phase there are three invariant subspaces of eigenvectors corresponding to three classes of eigenvalues. In this appendix we shall list the eigenvalues $\lambda_\lambda$ and $\lambda_Q$ and their multiplicity $\mu$ for each different subspace.

### 1. Longitudinal eigenvalues

For each diagonal sub-matrix there are two couples of longitudinal eigenvalues with multiplicity $\mu_L = 1$

$$\lambda_\Lambda^{LA} = \frac{A + D \pm |A - D|}{2} \qquad \lambda_Q^{LA} = f(q_1, \mathbf{k})$$

where

$$A = 1 - q_1^2 + 2(m-2)(q_1 - q_1^2) - \frac{(m-2)(m-3)}{2} C_{[abcd]} \tag{A1}$$

$$D = -1 + 2(m-1)q_1 + (m-1)^2 q_1^2 \tag{A2}$$

### 2. Anomalous eigenvalues

For each diagonal sub-matrix there are four couples of anomalous eigenvalues.

Two of them have multiplicity $\mu_A = (n-m)/m$ and coincide with the longitudinal ones.

The other two have multiplicity $\mu_A = n(m-1)/m$ and are

$$\lambda_\Lambda^{LA} = \frac{A' + D' \pm |A' - D'|}{2} \qquad \lambda_Q^{LA} = f(q_1, \mathbf{k})$$

where

$$A' = 1 - q_1^2 + (m-4)(q_1 - q_1^2) - (m-3)C_{[abcd]} \tag{A3}$$

$$D' = -1 + (m-2)q_1 - (m-1)q_1^2 \tag{A4}$$

### 3. Replicon eigenvalues

There are four couples of different replicon eigenvalues

$$\lambda_\Lambda^R = P_1 - 2Q_1 + R_0 \qquad \lambda_Q^R = f(q_1, \mathbf{k}), \qquad \mu_R = n\frac{(m-3)}{2} \tag{A5}$$

$$\lambda_\Lambda^R = P_0 + 2(m-1)Q_0 + (m-1)^2 q_1^2 \qquad \lambda_Q^R = 0, \qquad \mu_R = n\frac{(n-3m)}{2m^2} \tag{A6}$$

$$\lambda_\Lambda^R = P_0 + (m-2)Q_0 - (m-1)q_1^2 \qquad \lambda_Q^R = 0, \qquad \mu_R = n\frac{(n-2m)(m-1)}{m^2} \tag{A7}$$

$$\lambda_\Lambda^R = P_0 - 2Q_0 + q_1^2 \qquad \lambda_Q^R = 0, \qquad \mu_R = n\frac{(n-m)(m-1)^2}{2m^2} \tag{A8}$$

where

$$P_0 = -1 \qquad P_1 = q_1^2 - 1 \tag{A9}$$

$$Q_0 = -q_1 \qquad Q_1 = q_1^2 - q_1 \tag{A10}$$

$$R_0 = q_1^2 - C_{[abcd]} \tag{A11}$$

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